Exercise 14

Evaluate the line integral, where C is the given curve.

$$\int_C y\,dx + z\,dy + x\,dz, \quad C:\ x = \sqrt{t},\ y = t,\ z = t^2, \quad 1 \le t \le 4$$

Solution

Write the integral in terms of a dot product.

$$\int_C y\,dx + z\,dy + x\,dz = \int_C \langle y, z, x\rangle \cdot \langle dx, dy, dz\rangle$$

With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C y\,dx + z\,dy + x\,dz &= \int_1^4 \langle y(t), z(t), x(t)\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle dt \\
&= \int_1^4 \langle t, t^2, \sqrt{t}\rangle \cdot \left\langle \frac{1}{2}t^{-1/2}, 1, 2t\right\rangle dt \\
&= \int_1^4 \left(\frac{1}{2}t^{1/2} + t^2 + 2t^{3/2}\right) dt \\
&= \left(\frac{1}{3}t^{3/2} + \frac{1}{3}t^3 + \frac{4}{5}t^{5/2}\right)\Bigg|_1^4 \\
&= \frac{1}{3}(4^{3/2} - 1^{3/2}) + \frac{1}{3}(4^3 - 1^3) + \frac{4}{5}(4^{5/2} - 1^{5/2}) \\
&= \frac{722}{15}.
\end{aligned}$$